Abstract

I introduce strategic externalities into a standard herding model. It is assumed that such externalities only affect successors. I study the interplay of informational and strategic externalities and I determine how their relative magnitudes affects the occurrence of herds and informational cascades. If strategic externalities (measured by a parameter $\sigma$) are negative and sufficiently strong, an informational cascade arises but there is no herding (‘anti-herding’ occurs). This contrasts with the existing literature which generally finds that an informational cascade implies herding. In a continuous-signal version of the model, I show that there exists an interval of $\sigma$ in which learning is more efficient than in the $\sigma = 0$ case. Moreover, there always exists one value of $\sigma$ such that every individual reveals her signal. I make different assumptions on the observability of actions and I show that agents may engage in either imitative or contrarian behavior, depending on the value of $\sigma$. It is shown that some previous results on herding and informational cascades are not robust. Finally, I study the model under binary signals. It is shown that negative strategic externalities always prevent herding and may lead to a considerable increase in the efficiency of learning.

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Keywords: Social Learning, Strategic externalities, Herding, Informational Cascades


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1 Introduction

There is a large number of social and economic situations in which individuals are influenced by the actions of others. Common examples include consumer purchase decisions, the choice of a restaurant, the adoption of a new technology and asset market decisions. It is well known that individuals may underestimate their private information in such situations and that this can lead to inefficient behavior such as fads, booms, financial market bubbles and busts, bank runs, or the failure of firms to coordinate on the adoption the best technology. A comprehensive literature pioneered by Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992) has used herd behaviour and informational cascades to explain a variety of such phenomena. This literature analyzes sequential action models with incomplete and asymmetric information, where agents can observe actions taken by their predecessors but not the information upon which these actions were taken. These models generally predict that agents eventually herd on one action, i.e. that rational agents imitate their predecessors’ behavior even if it conflicts with their own information.

A central assumption of most herding models is that agents’ payoffs are independent of the actions of others. Therefore, agents are concerned about the actions of others only to the extent that they reveal private information about the unknown state of the world. In other words, the only externality present in these models is an informational one. In many real-world situations, however, social learning is significantly affected by the direct dependence of payoffs on actions. This dependence may have a crucial impact on the occurrence of herds and informational cascades. A long waiting line in front of a night club, for example, may well indicate high quality of the club, but herding nevertheless only occurs to a certain extent because eventually newcomers are deterred by the long line. This paper contributes to the literature on social learning by relaxing the assumption of pure informational externalities and by allowing for strategic externalities. Strategic externalities will be referred to as strategic substitutabilities (complementarities) if they are positive (negative).

In models with pure informational externalities, there is no incentive for agents to behave strategically. Therefore, they exhibit pure backward-looking behaviour, which is relatively easy to study. In the presence of strategic externalities, however, agents also need to consider the impact of their actions on the actions of successors, i.e. they need to exhibit forward-looking behavior. In the technology adoption problem, for example, the choices of predecessors are just as important as the choices of successors because each firm attempts to adopt the technology used by the majority of all firms. This may lead
to complex types of behavior, involving strategic signalling of private information. Agents’ behavior in such models is very hard to study and there is generally a multiplicity of equilibria, see for example Dasgupta (2000) or Drehmann, Oechssler and Roider (2007).

In this paper, I analyze strategic externalities in standard herding models under the restricting assumption that externalities are imposed only on successors. This implies that agents are entirely backward-looking. They do not behave strategically but are merely interested in learning their predecessors’ private information from previous actions, while taking into account the strategic externality effects of these actions. In appendix A, I briefly introduce a model with allows individuals to be backward-looking as well as forward-looking. Since externalities cancel each other out in this model, there exists a simple and efficient equilibrium in which agents rely entirely on their own signal.

There are many real-world situations in which individuals impose externalities only on their successors. Examples include waiting lists for a new product or the choice of a parking lot (if availability is not observable before entering the parking lot). More generally, all first-come-first-served queuing systems with waiting costs and uncertainty about the quality of service are valid examples, see Debo, Parlour and Rajan (2005). While these examples involve strategic substitutabilities, there are other situations in which actions impose strategic complementarities on successors. Consider, for example, an individual’s decision on whether or not to commit a crime. If a potential criminal observes that many peers have committed the same crime, she may infer that the probability of gain is high, see Kahan (1997). In addition to this informational externality, there is a strategic complementarity (imposed only on successors) due to reduced law enforcement in case many crimes have been committed in the past.

There is another way of interpreting the backward-looking approach. The model adopted here is equivalent to a model in which payoffs are fully dependent on actions of predecessors and successors, but in which individuals are boundedly rational in the sense that they do not consider the impact of their choice on the actions of successors. Experimental evidence suggests that individuals may indeed behave in this fashion, see Drehmann, Oechssler and Roider (2007).

The general model is introduced in section 2. I consider a standard setting of social learning, in which agents choose binary actions in a chronological order. Before deciding on a particular action, each agent receives some information (this will be specified below) about the behavior of her predecessors and a private signal that is correlated with the unknown state of the world. The action space is discrete throughout this paper, and I study the model
under both a continuous and a discrete signal space in order to make it comparable to large parts of the social learning literature. While the bulk of this literature is of the discrete-signal-discrete-action type, some interesting models of the continuous-signal-discrete-action type have been proposed as well, most notably Smith and Sørensen (2000).

The main focus of my paper is on the efficiency of learning. While sufficiently strong strategic substitutabilities always imply inefficient and strongly non-conformist behavior (so-called anti-herding), I show that moderate strategic substitutabilities may lead to an increase in efficiency since neither herding nor anti-herding arise. In particular, I show that there exists a special case in which the informational externality of each action and the corresponding strategic externality cancel each other out. This induces agents to rely entirely on their private information and therefore learning is efficient. Moreover, I show that strategic complementarities have a negative effect on the efficiency of learning since they increase agents’ tendency towards conformity and therefore reinforce inefficient herding.

In section 3, I study the model under a continuous signal space. Continuous signals have the feature that they allow a distinction between strong and weak signals, which affects agents’ tendency towards incorporating predecessor’s decisions into their own action choice. It will turn out that analyzing this tendency subject to agents’ confidence in their private information reveals some interesting insights. I show how agents’ signals influence their decisions and under what conditions (anti-)herding and cascade behavior arises. Most analytical expressions are derived under a uniform distribution. However, at the end of the section, I show that the results do not change qualitatively if one allows for a general distribution.

The two central concepts studied here are herds and informational cascades. While in discrete models such as Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992), these notions are used to address the same phenomenon, Smith and Sørensen (2000) introduce them as two separate concepts. According to their definitions, an informational cascade is an infinite sequence of agents who all neglect their private information and base their actions entirely on the observations of the behavior of others. Hence, once a cascade starts, public information stops accumulating. Herd behavior, on the other hand, describes a situation in which all agents take the same action, not necessarily ignoring their private information. In this case, public information can in principle still be accumulated.

The social learning literature generally finds that informational cascades arise in situations in which agents imitate their predecessors, i.e. they are due to conformist behavior. This paper shows that in the presence of strategic externalities, extreme forms of non-conformist behavior may provide another
explanation for informational cascades. It is shown that if strategic substitutabilities are sufficiently strong, agents neglect their private information and engage in anti-herding behavior. In other words, I show that in the presence of strategic externalities, informational cascades are neither necessary nor sufficient for the occurrence of herding behavior. This result contrasts with previous results obtained in models of pure informational externalities.

A common feature of most models of social learning is that agents are able to observe all actions taken by their predecessors (which I refer to as ‘perfect observability’). In reality, however, individuals can usually only observe a limited number of others. Several models of social learning take this into account by making restrictions on the observability of other players. For example, Smith and Sørensen (1996) study a model in which recent predecessors are more likely to be observed. In section 3, I study two specifications of the continuous-signal version of my model; one with perfect observability and one with imperfect observability. For the sake of simplicity, I assume that under imperfect observability, agents only observe the actions taken by their immediate predecessors. Both of these informational scenarios are treated by Celen and Kariv (2004). They find that the informational structure matters for the occurrence of herds. I show that this result only holds true in the presence of pure informational externalities. In the presence of strategic externalities, herding and cascade behavior occur independently of the informational scenario. Moreover, I show that some other results of Celen and Kariv (2004) are not robust since they strongly depend on the absence of strategic externalities.

In addition to greater realism, the imperfect observability scenario has another motivation. Since each agent’s information structure is relatively simple in this case (it is two-dimensional: agents observe their private signal and their predecessor’s action), it is possible to study directly how agents respond to behavior of their predecessors depending on the degree of strategic externalities. In particular, imitative and contrarian behavior can be elicited (contrarian behavior is defined as taking the action opposite to the predecessor’s action). In the perfect observability scenario, this is analytically impossible due to the large and time-varying sets of possible observational histories. I find that if agents only observes their predecessors’ actions, weak signals induce them to (rationally) base their action choice on the observation of their predecessor’s action, by entirely neglecting their own signal. This either happens through imitating or through contrarian behavior. If strategic externalities are strong, even agents with relatively strong private signals engage into imitation or contrarian behavior and, at the extreme, this may lead to herding or anti-herding. This analysis produces (recursive) analytical expressions describing the relationship between individual tenden-
cies towards imitative or contrarian behavior and the overall social tendency towards herding, anti-herding or cascade behavior.

In section 4, I study the model under the assumption of binary signals (and perfect observability). I show that in the presence of strategic complementarities, herding always occurs, as in Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992). Under strategic substitutabilities, herding never occurs but there always is an informational cascade with anti-herding. It is shown that anti-herding typically occurs after a substantial degree of learning has taken place and I show that strategic complementarities may significantly increase the efficiency of learning. However, it is shown that there always exist two possible outcomes of social learning, one in which the public belief is close to the true state of the world and one, in which the public belief assigns a probability close zero to it. It is shown that the problem of convergence to either one of these two states can be transformed into a version of the Gambler’s Ruin Problem. Specifically, the probability that social learning is correct is equivalent to the probability that the gambler doubles her fortune, while the probability that social learning is wrong is equal to the probability that the gambler goes broke. Using the solution to the Gambler’s Ruin Problem, it is shown that social learning becomes arbitrarily accurate (i.e. the public belief is arbitrarily close to the true state of the world, with arbitrarily high probability), provided that either the precision of the signal goes to one or the magnitude of the strategic externalities goes to zero. However, there may be very long spells of imitative behavior between two agents who reveal their signals. I show via Monte Carlo simulation that learning may indeed take a very long time to reach the absorbing state.

Some alternative studies of the interplay between informational and strategic externalities have been proposed. Choi (1997) analyzes this relationship in a technology adoption framework. In his model, firms do not observe private signals about the quality of alternative technologies and therefore herding is merely due to strategic externalities. Dasgupta (2000) introduces strategic complementarities into a sequential choice model with continuous private signals. Agents receive a positive return from investing only if all the other agents also choose to invest. Moreover, Frisell (2003) studies the interplay between strategic and informational externalities in a waiting game. Finally, there exists a recent literature that combines models of social learning with queueing models, see Debo, Parlour and Rajan (2005) and Debo and Veeraraghavan (2008). This literature analyzes the relationship between the (positive) informational externalities of long lines and the (negative) strategic externalities induced by the corresponding waiting costs.
2 The Basic Model

Consider a countable set of identical agents, indexed by \( n = 1, 2, ..., N \), where \( N \in \mathbb{N} \cup \infty \). Each agent \( n \) has to make a once-in-a-lifetime decision \( a_n \) between two alternatives from the set \( A \equiv \{-1, 1\} \). Decisions are made sequentially in an exogenously determined order. Let \( N_a(n) \) denote the number of predecessors of agent \( n \) who have chosen alternative \( a \), i.e. \( N_a(n) = \sum_{i=1}^{n-1} 1_{\{a_i = a\}} \), where \( 1_{\{\cdot\}} \) represents the indicator function. The payoff of alternative \( a \) depends on \( N_a \) as well as on the payoff-relevant quality \( Q_a \subset \mathbb{R} \) of alternative \( a \). Since relative qualities are sufficient to describe agent’s choices, the state of the world is defined as \( \theta_0 = Q_1 - Q_{-1} \). Assume that agents are risk-neutral with utility function

\[
u_n(a) = \sigma N_a(n) + Q_a(\theta_0),\]

where \( \sigma \in (-\infty, +\infty) \) measures the size of strategic externalities.

The state of the world is unknown to all agents but each agent \( n \) observes a private signal \( \theta_n \in \Theta_S \subset \mathbb{R} \) which is correlated with \( \theta_0 \). Assume that, conditional on \( \theta_0 \), all \( \theta_n \) are iid and follow a commonly known distribution.

Furthermore, let \( I_n \) denote the payoff-relevant information available to agent \( n \). I study two different informational scenarios. In the perfect observability (PO) scenario, agents observe the actions of all their predecessors, whereas in the imperfect observability (IO) scenario only the immediate predecessor’s action is observable. Formally, \( I_n^{PO} = (\theta_n, (a_i)_{i=1}^{n-1}) \) and \( I_n^{IO} = (\theta_n, a_{n-1}) \). It follows that for given \( I_n \), agent \( n \)’s optimal decision rule is given by:

\[
a_n = 1 \text{ if and only if } E \left[ \sigma N_1(n) - \sigma N_{-1}(n) + Q_1(\theta_0) - Q_{-1}(\theta_0) \mid I_n \right] \\
= E \left[ \sigma \sum_{i=1}^{n-1} a_i + \theta_0 \mid I_n \right] \geq 0. \quad (1)
\]

The decision rule contains the implicit assumption that agents always chose alternative \( a = 1 \) in case of indifference. This assumption does not affect the results since indifference is an event of zero-measure for each of the specifications of the model.

The focus of this paper is mainly on the occurrence of informational cascades, herds and antiherds (these concepts will be defined below), i.e. on agents’ behavior as the number \( N \) of agents goes to infinity. Therefore, I either consider the limit of a sequence of economies indexed by \( N \), where \( N \) tends to infinity (as in section 3) or I set \( N \) directly equal to infinity (as
in section 4). The finite-economy approach is used in the continuous signal case in section 3 because under this specification of the model, an economy of finite (but possibly very large) size is required. Note that this approach is analytically straightforward since agents’ behavior exclusively depends on the actions of their predecessors and therefore the size of the economy is irrelevant to the decision of any agent. For notational convenience, I occasionally consider only the limit as \( n \) goes to infinity without explicitly stating the double limit as both \( N \) and \( n \) tend to infinity.

**Definition 1.** An informational cascade occurs if there is some agent \( n_0 \) and some alternative \( a \in \mathcal{A} \) such that for all \( n \geq n_0 \): \( \theta_n \in \Theta_S \Rightarrow a_n = a \) (i.e. all agents \( n \geq n_0 \) disregard their private signals). In case \( N < \infty \), this property has to be satisfied in the limit as \( N \) goes to infinity.

**Definition 2.** A run\(^1\) (anti-run) at position \( n_0 \) of length \( m^N_{n_0} \) satisfies \( a_n = a_{n-1} \) (\( a_n = -a_{n-1} \)) for all \( n = n_0 + 1, n_0 + 2, ..., n_0 + m^N_{n_0} \). If \( N = \infty \), a herd (anti-herd) is said to occur if there is some \( n_0 \) such that \( m^N_{n_0} = \infty \). If \( N < \infty \) a herd (anti-herd) is said to occur if there exists some \( n_0 \) such that \( \lim_{N \to \infty} m^N_{n_0} = \infty \) (i.e. a run (anti-run) that grows arbitrarily large as the size of the economy goes to infinity). Herding (anti-herding) is complete if \( n_0 = 1 \).

**Definition 3.** Agent \( n \) is self-reliant (for given \( \theta_n \)), if \( a_n \) is independent of \((a_i)_{i=1}^{n-1}\). Agent \( n \) is purely self-reliant if she is self-reliant for all \( \theta_n \in \Theta_S \).

## 3 Continuous signals

In this section, private signals \( \theta_n \) follow a symmetric distribution with c.d.f. \( F \) over the support \( \Theta_S \equiv [-b, b] \), where \( b < \infty \). For simplicity, I assume that \( F \) is continuous and differentiable. The set of agents is finite, i.e. \( N < \infty \), and the state of the world is given by the sum of individual signals, i.e. \( \theta_0 = \sum_{i=1}^{N} \theta_i \). Hence, agent \( n \)’s optimal behavior is determined by:

\[
a_n = 1 \text{ if and only if } E \left[ \sigma \sum_{i=1}^{n-1} a_i + \sum_{i=1}^{N} \theta_i \mid I_n \right] \geq 0
\]

Using the fact that the expected value of successors’ signals is always zero, the decision rule can conveniently be expressed as a history-contingent cutoff-rule for \( \theta_n \):

\[
a_n = 1 \text{ if and only if } \theta_n \geq -E \left[ \sigma \sum_{i=1}^{n-1} a_i + \sum_{i=1}^{n-1} \theta_i \mid I_n \right]. \tag{2}
\]

\(^1\)This term is borrowed from Drehmann, Oechssler and Roider (2007).
Let \( \hat{\theta}_n \) denote the right-hand side of this inequality, i.e. the cutoff-level for agent \( n \)'s signal. Note that \( \hat{\theta}_n \) is sufficient to describe the behavior of agent \( n \). Therefore, I focus my analysis on the stochastic process \( \{\hat{\theta}_n\} \), which I will refer to as the learning process. In the remainder of this section, I study the properties of the learning process under different assumptions on the signal distribution \( F \) as well as different assumptions on the degree of observability of other agents’ actions. In subsection 3.1, I assume that \( F \) is uniform, while subsection 3.2 shows that this assumption can be relaxed without losing some of the key properties.

### 3.1 Uniform Signals

Let \( F \) be a uniform distribution on the interval \([-1, 1]\), i.e. \( b = 1 \). In the following, I study the corresponding learning processes in the perfect and the imperfect observability scenarios.

#### 3.1.1 Perfect Observability and Uniform Signals

Let \( I_n = I_{nPO} \), i.e. assume that in addition to her private signal \( \theta_n \), each agent \( n \) observes the entire sequence \((a_i)_{i=1}^{n-1}\) of predecessors’ actions before taking a decision. In this case, the learning process is a stochastic process which is characterized as follows.

**Proposition 1.** The learning process satisfies \( \hat{\theta}_1 = 0 \) and

\[
\hat{\theta}_{n+1} = \begin{cases} 
1 & \text{if } \hat{\theta}'_{n+1} > 1 \\
-1 & \text{if } \hat{\theta}'_{n+1} < -1 \\
\hat{\theta}'_{n+1} & \text{otherwise}
\end{cases}
\]

where \( \hat{\theta}'_{n+1} = \frac{\theta_n}{2} - a_n \left(\frac{1}{2} + \sigma\right) \).

**Proof.** Using the fact that \( F \) is uniform and equation (2), it follows that

\[
a_n = 1 \text{ iff } \theta_n \geq \hat{\theta}_n = -\mathbb{E} \left[ \sum_{i=1}^{n-1} (\theta_i + \sigma a_i) \mid (a_i)_{i=1}^{n-1} \right] = \hat{\theta}_{n-1} - \mathbb{E} \left[ \theta_{n-1} + \sigma a_{n-1} \mid (a_i)_{i=1}^{n-1} \right] = \hat{\theta}_{n-1} - \left\{ \begin{array}{ll} \frac{1}{2}(\hat{\theta}_{n-1} + 1) + \sigma & \text{if } a_{n-1} = 1 \\ \frac{1}{2}(\hat{\theta}_{n-1} - 1) - \sigma & \text{if } a_{n-1} = -1 \end{array} \right. = \hat{\theta}_{n-1} - a_{n-1} \left(\frac{1}{2} + \sigma\right) .
\]

Clearly, agent one’s cutoff is \( \hat{\theta}_1 = 0 \), because \( a_1 = 1 \) iff \( \theta_1 \geq 0 \). \( \square \)
Figures 1 and 2 show the realizations of several Monte Carlo simulations of \( \hat{\theta}_n \) for different \( \sigma \) (and for sufficiently large \( N \)). In case \( \sigma = 0.01 \), the learning process eventually settles down in the point \( \hat{\theta} = 1 \), which implies identical actions, regardless of private signals. This is equivalent to an informational cascade with herding. Moreover, the plots suggest that the autocorrelation of the learning process increases in \( \sigma \). Intuitively, this is due to the fact that increasingly strong strategic substitutabilities increase agents’ tendency to ‘avoid’ the actions of their predecessors. The following proposition describes the behavior of the learning process for different \( \sigma \).

Figure 1: Simulation runs of \( \hat{\theta}_n \) for \( \sigma = 0.01 \) (left) and \( \sigma = -0.01 \) (right).

Figure 2: Simulation runs of \( \hat{\theta}_n \) for \( \sigma = -0.1 \) (left) and \( \sigma = -1 \) (right).
Proposition 2.

(a) If $\sigma > 0$, an informational cascade with herding almost surely occurs after finite time.

(b) If $\sigma = 0$, all agents are self-reliant with positive probability and there is no informational cascade but herding occurs after a finite time, see Çelen and Kariv (2004).

(c) If $\sigma \in (-2, 0)$, all agents are self-reliant with positive probability and neither an informational cascade nor herding occurs. In particular, if $\sigma = -0.5$, all agents are purely self-reliant.

(d) If $\sigma \leq -2$, an informational cascade with anti-herding occurs.

Proof. All claims follow directly from proposition 2 in section 3.2, with the exception of part (c). Let $\sigma \in (-2, 0)$. The boundaries of the learning process are given by its fixed points. The solution to the equation $\hat{\theta} = \hat{\theta} / 2 - a_n(0.5 + \sigma)$ is either $\hat{\theta} = -(1 + 2\sigma)$ or $\hat{\theta} = (1 + 2\sigma)$. Hence, $\hat{\theta}_n \in (-1 - 2\sigma, 1 + 2\sigma) \subset [-1, 1]$ and therefore all agents are self-reliant with positive probability.

Consider the case $\sigma \in (-0.5, 0)$. It is shown in the proof of proposition 2 that in this case $\hat{\theta}_n \in (-1 + 2\sigma, 1 + 2\sigma) \subset [-1, 1]$ for all $n$. In particular, if $\sigma$ is close to 0, the cutoff process is close to either 1 or $-1$ and switches occur only rarely (see figure 1, right). Therefore, agent’s behavior is characterized by long spells of imitative behavior and only rarely (with probability close to $\sigma$) an agent receives a signal which is sufficiently strong to induce her to deviate from the previous run. By overturning a previous run, the deviator reveals an extreme contrary signal which induces a big jump in the cutoff process (see figure 1, right) and makes her successor close to indifferent between the two actions. It is easy to see that in the limit as $N$ and $n$ tend to infinity, the learning process never settles down because behavior overturns forever. This result contrasts with Çelen and Kariv (2004) who demonstrate that in the absence of strategic externalities, i.e. $\sigma = 0$, a herd must arise in finite time (intuitively, their result is based on the fact that the probability of overturning an ongoing run goes to zero quite rapidly). Hence, their finding is not robust in the sense that every arbitrarily small size of strategic substitutability causes behavior to overturn forever and therefore makes herding impossible.

An interesting phenomenon arises if $\sigma = -0.5$. In this case, strategic and informational externalities cancel each other out. Assume that some agent $k$ observes the action $a_n$ of some other agent $n \neq k$. This observation reveals that agent $n$’s private signal has been in favor of alternative $a_n$, which has a
positive effect on agent $k$’s expected payoff of alternative $a$. In case $\sigma = -0.5$, the strategic substitututability imposed by this choice has an opposing effect of exactly the same magnitude and therefore the payoff of alternative $a$ remains unaffected by the observation of agent $n$’s choice. Therefore, despite being informative about the true state of the world, the observations of the behavior of others are worthless in this case and agents rely entirely on their private signals.

3.1.2 Imperfect Observability and Uniform Signals

In addition to her private signal $\theta_n$, agent $n$ observes only her immediate predecessor’s action $a_{n-1}$ before taking an action. The following proposition characterizes the corresponding learning process in terms of two cutoff levels, $\underline{\theta}$, and $\overline{\theta}$, depending on whether $a_{n-1} = -1$ or $a_{n-1} = 1$.

**Proposition 3.** The learning process satisfies $\hat{\theta}_1 = 0$ and

$$\hat{\theta}_{n+1} = \begin{cases} 
\overline{\theta}_{n+1} & \text{if } a_n = 1 \\
\underline{\theta}_{n+1} & \text{if } a_n = -1 
\end{cases}$$

where $\underline{\theta}_{n+1} = -\overline{\theta}_{n+1}$,

$$\overline{\theta}_{n+1} = \begin{cases} 
1 & \text{if } \overline{\theta}_{n+1} > 1 \\
-1 & \text{if } \overline{\theta}_{n+1} < -1 \\
\overline{\theta}_{n+1} & \text{otherwise}
\end{cases}$$

and $\underline{\theta}_{n+1} = -\sigma - \frac{1}{2} \left(1 + \theta_n^2\right)$.

**Proof.** See Appendix.

Figure 3 illustrates the evolution of the cutoff levels $\underline{\theta}$ and $\overline{\theta}$ as specified in proposition 3. In case $\sigma > -0.5$ (left graph), $\underline{\theta}_n$ is larger than $\overline{\theta}_n$. Hence, whenever agent $n$ receives a private signal within the interval $(\underline{\theta}_n, \overline{\theta}_n)$, she chooses the same action as her predecessor. This is the case because her private signal is too close to zero in order to be sufficiently informative to guide her decision. If, on the other hand, $\sigma < -0.5$ (right graph), $\underline{\theta}_n$ is smaller than $\overline{\theta}_n$ and whenever their private signal is within the interval $(\overline{\theta}_n, \underline{\theta}_n)$, agents choose the action opposite to their predecessor’s action. This leads to the following definition.

**Definition 4.** Agent $n$ is a conformist if $\overline{\theta}_n < \underline{\theta}_n$ and $\theta_n \in [\underline{\theta}_n, \overline{\theta}_n]$, and a contrarian if $\overline{\theta}_n > \underline{\theta}_n$ and $\theta_n \in [\overline{\theta}_n, \underline{\theta}_n]$. Agent $n$ is a pure conformist (contrarian) if she is a conformist (contrarian) for all $\theta_n \in \Theta_S$.  

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Figure 3 suggests that in case $\sigma \in (-2,0)$, all cutoff values $\theta_n$ and $\bar{\theta}_n$ are strictly within the interval $(-1,1)$ (with the exception of agent $n = 1$ in case $\sigma$ is close to $-2$) and therefore all agents are self-reliant with positive probability, provided that their private signals are sufficiently close to either 1 or -1. Hence, an informational cascade does not arise in this case. As $\sigma$ gets close to either $-2$ or 0, conformist, resp. contrarian, behavior becomes increasingly likely because the interval $[\theta_n, \bar{\theta}_n]$, resp. $[\bar{\theta}_n, \theta_n]$, approaches the entire signal space. Beyond these thresholds, agents always disregard their private information and select their optimal action entirely on the grounds of their predecessor’s action, i.e. agents are pure conformists in the former case and pure contrarians in the latter. An informational cascade always arises in this case due to herding, resp. anti-herding. Note that, like under perfect observability, strategic and informational externalities cancel each other out if $\sigma = -0.5$. It is easy to see that behavior must be independent of the observational structure in this case. The following proposition summarizes.

**Proposition 4.**

(a) If $\sigma < -0.5$ ($\sigma > -0.5$), all agents are contrarians (conformists) with positive probability and there are no conformists (contrarians).

(b) If $\sigma > 0$, there exists some agent $n_0 \geq 2$ such that all agents $n \geq n_0$ are pure conformists and an informational cascade with herding occurs.

(c) If $\sigma \in (-2,0]$, all agents are self-reliant with positive probability and neither an informational cascade nor herding occurs. In particular, if $\sigma = -0.5$, all agents are purely self-reliant.
If $\sigma \leq -2$, all agents are pure contrarians and an informational cascade with anti-herding occurs.

Proof. Let me use the fact that $\theta_n = -\theta_n$ for all $n$ (see proposition 3).

part (a): Let $x := -\sigma - 0.5 > 0$ and let $x_m := \min\{x, 1\}$. First, I show that $\theta_n > 0 \forall n \geq 2$. Assume that $\theta_n \in (0, x_m)$. This implies that $\theta_{n+1} = \min\{x - \theta_n^2/2, 1\} \in (0, x_m)$ and the claim follows by induction since $\theta_1 = 0$, $\theta_2 = x_m$ and $\theta_3 = \min\{x - x^2/2, 1\} \in (0, x_m)$. The second part of the proof works in the same way.

part (b): See the general proof of proposition 9.

part (c): In case $\sigma \in (-2, 0)$, it can be shown (cf proof of proposition 5) that $\theta_n \to -1 + \sqrt{-2\sigma}$ and $\theta_n \to 1 - \sqrt{-2\sigma}$ as $N \to \infty$ and $n \to \infty$. Hence all agents are self-reliant with positive probability. In order to show that all agents are purely self-reliant if $\sigma = -0.5$, it is sufficient to show that $\theta_n = 0 \forall n$. This follows by induction since $\theta_n = 0$ implies that $\theta_{n+1} = -\sigma - 1/2 = 0 \forall n$ and since $\theta_1 = 0$. In the special case $\sigma = 0$, the interval $[\theta_n, \theta_n]$ converges to the interval $[-1, 1]$. Nevertheless, a herd does not exist as shown by Çelen and Kariv (2004).

part (d): It can be shown via direct calculation that in case $\sigma \leq -2$, $\theta_n = 1$ for all $n \geq 2$ which implies an informational cascade with anti-herding.

The question of whether herding occurs in the limiting case without strategic externalities is answered by Çelen and Kariv (2001). In this case, $\theta_n$ monotonically increases in $n$ and the interval $[\theta_n, \theta_n]$ converges to the entire signal space $[-1, 1]$ (in the limit as $N$ and $n$ tend to infinity). Çelen and Kariv (2001) show that the cutoff process $\{\hat{\theta}_n\}$ as defined in proposition 3 does not converge in this case. Since divergence of cutoffs implies divergence of actions, standard herd behavior is impossible, even though the expected length of runs is increasing and goes to infinity. In contrast to this result, it is shown here that if $\sigma > 0$, the interval $[\theta_n, \theta_n]$ reaches the entire signal space after some finite number of agents and therefore an informational cascade and herd behavior arise after finite time. Since this is the case for any arbitrarily small size of strategic complementarities, the above analysis shows that the no-herding result of Çelen and Kariv (2001) is not robust in this sense.

As shown by the proposition, if strategic externalities are within the interval

\footnote{For any agent $k$, it can be shown that, as $N \to \infty$, the probability that all of agent $k$’s successors take the same action is equal to zero, i.e. $\Pi_{n=k}^{\infty} \frac{1-\eta}{2^n} = 0$, see corollary 5 in Çelen and Kariv (2001).}
(-2,0), neither an informational cascade nor (anti-)herding occur because agents are self-reliant with strictly positive probability. As noted before, agents become increasingly likely to engage in conformist or contrarian behavior if \( \sigma \) approaches the boundaries of this interval and therefore one expects behavior to come close to cascade and (anti-)herding behavior at these boundaries. The following proposition supports this expectation with analytical expressions. In particular, the expected length of a run is calculated as a function of \( \sigma \). This expression tends to 0 as \( \sigma \) approaches \(-2\) (from above) due to an increasing tendency towards anti-herding. As \( \sigma \) approaches 0 (from below), the expected length of a run tends to \(+\infty\) and hence approaches herding behavior in the limit (see figure 4). Similarly, the proposition shows that behavior approaches cascade behavior at the boundaries in the sense that the expected number of consecutive contrarians (conformists) approaches \(+\infty\) as \( \sigma \to -2 \) (as \( \sigma \to 0 \)). Moreover, as \( \sigma \) approaches \(-0.5\), runs become increasingly short and entirely disappear in case \( \sigma = -0.5 \) since all agents are entirely self-reliant (see figure 4).

![Figure 4: Behavior approaches cascade and (anti-)herding behavior as \( \sigma \) approaches the boundaries of the interval \([-2,0]\).](image)

**Proposition 5.**

(a) If \( \sigma \in (-2,0) \), \( \theta_n \to 1 - \sqrt{-2\sigma} \) as \( n \to \infty \) and the expected length of a run starting at agent \( n \) tends to \( \frac{2}{\sqrt{-2\sigma}} - 1 \).

(b) In case \( \sigma \in (-2,-0.5] \), the expected number of consecutive contrarians following agent \( n \) tends to \( \frac{\sqrt{-2\sigma} - 1}{2 - \sqrt{-2\sigma}} \) as \( n \to \infty \). In case \( \sigma \in (-0.5,0) \) the expected number of consecutive conformists following agent \( n \) tends to \( \frac{1}{\sqrt{-2\sigma}} - 1 \) as \( n \to \infty \).
Proof. part (a): The limit point $1 - \sqrt{-2\sigma}$ of the sequence $\{\theta_n\}$ is given by the (feasible) root of the equation $\theta_{n+1} = \theta_n = 0$. Furthermore, the probability for agent $n$ to choose action $a_n = a_{n-1}$ is equal to $\Pr(\theta_n \in (\bar{\theta}_n, 1)) = (1 - \bar{\theta}_n)/2$ and approaches $1 - \sqrt{-2\sigma}/2$ as $n \to \infty$. Hence, the distribution of the length of a run following agent $n$ approaches a geometric distribution with parameter $p = \sqrt{-2\sigma}/2$ and expected value $(1 - p)/p = 2\sqrt{-2\sigma} - 1$.

part(b): In case $\sigma \in (-2, -0.5]$, the probability that agent $n$ is a contrarian is equal to $(\bar{\theta}_n - \theta_n)/2$ and approaches $\sqrt{-2\sigma} - 1$ as $n \to \infty$. Hence, the distribution of the number of successive contrarians approaches a geometric distribution with parameter $p = 2 - \sqrt{-2\sigma}$ and expected value $(1 - p)/p = \frac{\sqrt{-2\sigma} - 1}{2 - \sqrt{-2\sigma}}$. Similarly, in case $\sigma \in (-0.5, 0)$, the number of successive conformists approaches a geometric distribution with parameter $p = \sqrt{-2\sigma}$ and expected value $(1 - p)/p = \frac{1}{\sqrt{-2\sigma}} - 1$. \qed

Figure 5 summarizes the findings of this section. Remember from definition 2 that herding and anti-herding are complete if all agents $n \neq 1$ engage in it. Anti-herding is always complete and it is easy to show that complete herding occurs beyond $\sigma = 0.5$.

![Figure 5: The relationship between informational cascades, (anti-)herds and the size of strategic complementarities.](image)

Propositions 2 and 4 suggest that the informational structure does not affect the long-run outcome of behavior whereas strategic externalities do. Only in the absence of strategic externalities, the long-run outcome is affected by the informational structure. This shows that the main finding of Çelen and Kariv (2004) strongly depends on the absence of strategic externalities. The following corollary summarizes.

**Corollary 1.** For any $\sigma \neq 0$, an informational cascade (a herd, an anti-herd) occurs under perfect observability if and only if it occurs under imperfect observability. This property is violated in case $\sigma = 0$.  

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3.2 General Signals

In this section, I drop the assumption of a uniform signal distribution and allow for a general symmetric distribution characterized by its continuous and differentiable c.d.f. $F$ over the finite support $\Theta_S \equiv [-b, b]$. Let $E^+(x)$ denote the expected value of the signal $\theta$, conditional on being larger than some number $x$, i.e. $E^+(x) = \int_x^b \theta dF(\theta)$.

3.2.1 Perfect Observability and General Signals

Assume perfect observability, i.e. $I_n = I_n^{PO}$. Proposition 6 specifies a law of motion of the learning process and proposition 7 describes its properties. The most important properties of the uniform case carry over to the general case, such as the existence of a $\sigma$ under which both types of externalities can cancel each other out, the occurrence of an informational cascade with herding in the presence of positive $\sigma$ as well as the occurrence of an informational cascade with anti-herding in the presence of negative (and sufficiently small) $\sigma$. The main difference is that under a general signal distribution, some agents may be pure imitators or pure contrarians even if strategic externalities are moderate, i.e. $\sigma \in (-2b, 0)$. However, proposition 7 shows that an informational cascade or (anti-)herding is nevertheless impossible in this case.

**Proposition 6.** The learning process satisfies $\hat{\theta}_1 = 0$ and

$$
\hat{\theta}_{n+1} = \begin{cases} 
    b & \text{if } \hat{\theta}'_{n+1} > b \\
    -b & \text{if } \hat{\theta}'_{n+1} < -b \\
    \hat{\theta}'_{n+1} & \text{otherwise}
\end{cases}
$$

where $\hat{\theta}'_{n+1} = \hat{\theta}_n - a_n \left[ E^+(a_n \cdot \hat{\theta}_n) + \sigma \right]$.

**Proof.** Since all the information revealed by the history $(a_i)_{i=1}^{n-1}$ is already contained in $\hat{\theta}_n$, $\hat{\theta}_{n+1}$ is altered only by the new information revealed through $a_n$, i.e. $\hat{\theta}_{n+1} = \hat{\theta}_n - E\left(\theta_n + \sigma a_n | \hat{\theta}_n, a_n\right)$. The claim follows from the fact that $E\left(\theta_n | \hat{\theta}_n, 1\right) = E^+(\hat{\theta}_n)$ and $E\left(\theta_n | \hat{\theta}_n, -1\right) = -E^+(\hat{\theta}_n)$.

**Proposition 7.**

(a) If $\sigma > 0$, an informational cascade with herding almost surely occurs after finite time.
(b) If $\sigma = 0$, all agents are self-reliant with positive probability and there is no informational cascade but herding occurs after a finite time, see Smith and Sorenson (2000).

(c) If $\sigma \in (-2b, 0)$, neither an informational cascade nor herding occurs. If $\sigma = -E^+(0)$, all agents are purely self-reliant.

(d) If $\sigma \leq -2b$, an informational cascade with anti-herding occurs.

Proof. part (a): Consider any agent $n_0$ and let $\hat{\theta}_{n_0} \geq 0$. For $n' = \lceil b\sigma \rceil$, the probability that $\hat{\theta}_{n_0+n'} = b$ is at least $p = \left(\frac{1}{2}\right)^{n'}$. Since the symmetric argument applies for $\hat{\theta}_{n_0} < 0$, the claim follows from

$$\Pr(\min\{n : \hat{\theta}_n \in \{-b, b\}\} = \infty) \leq \lim_{i \to \infty} (1 - p)^i = 0.$$ 

part (c): It is sufficient to prove that for any agent $n_0$, there exists some agent $n \geq n_0$ such that $\hat{\theta}_n \in (-b, b)$. Consider some $n \geq n_0$ and suppose that $\hat{\theta}_n \notin (-b, b)$, i.e. either $\hat{\theta}_n = -b$ or $\hat{\theta}_n = b$. In this case either $\hat{\theta}_{n+1} = -b - \sigma \in (-b, b)$ or $\hat{\theta}_{n+1} = b + \sigma \in (-b, b)$, respectively. Moreover, $\sigma = -E^+(0)$ implies $\hat{\theta}_n = 0$ for all $n$.

part (d): It is easy to show that in case $\sigma \leq -2b$, $\hat{\theta}_2 \in \{-b, b\}$ and $\hat{\theta}_{n+1} = -\hat{\theta}_n$ for all $n \geq 2$, which implies the claim.

3.2.2 Imperfect Observability and General Signals

Assume imperfect observability, i.e. $I_n = I^O_n$. Proposition 8 specifies a law of motion of the learning process and proposition 9 describes its properties. Like in the perfect observability case, the most important properties of the uniform case carry over to the general case.

**Proposition 8.** The learning process satisfies $\hat{\theta}_1 = 0$ and

$$\hat{\theta}_{n+1} = \begin{cases} b & \text{if } \hat{\theta}_{n+1} > b \\ -b & \text{if } \hat{\theta}_{n+1} < -b \\ \hat{\theta}_{n+1} & \text{otherwise} \end{cases}$$

where $\hat{\sigma}_{n+1} = \hat{\theta}_n - 2F(\theta_n)[\theta_n + E^+(-\theta_n)] - \sigma$.

Proof. If can be shown that even in the case of a general signal distribution $P(a_n = 1) = 0.5$ and $\overline{\theta}_n + \overline{\theta}_n = 0$ is satisfied for all $n$ (see Çelen and Kariv (2001) for a similar proof). Moreover,

$$\overline{\theta}_{n+1} = [1 - F(\overline{\theta}_n)] [\overline{\theta}_n - E^+(\overline{\theta}_n) - \sigma] + [1 - F(\overline{\theta}_n)] [\overline{\theta}_n - E^+(\overline{\theta}_n) - \sigma]$$

$$= \overline{\theta}_n - 2 [F(\overline{\theta}_n)\overline{\theta}_n + (1 - F(\overline{\theta}_n))E^+(\overline{\theta}_n)] - \sigma. \quad (3)$$
The last transformation uses the fact that $1 - F(\theta_n) = F(-\theta_n) = F(\theta_n)$. Finally, the symmetry of the signal distribution implies that

$$(1 - F(\theta)) E^+(\theta) = F(\theta) E^+(-\theta).$$

Hence,

$$\theta_{n+1} = \theta_n - 2F(\theta_n) \left[ \theta_n + E^+(-\theta_n) \right] - \sigma. \quad (4)$$

**Proposition 9.**

(a) If $\sigma > 0$, there exists some agent $n_0 \geq 2$ such that all agents $n \geq n_0$ are pure conformists and an informational cascade with herding occurs. Herding is complete if $\sigma \geq b - E^+(0)$.

(b) If $\sigma \in [-b, 0]$, all agents are self-reliant with positive probability. There is no informational cascade and no (anti-)herding. If $\sigma = -E^+(0)$, all agents are purely self-reliant.

(c) If $\sigma < -b$, all agents are either contrarians or self-reliant. If $|\sigma|$ is sufficiently large, all agents $n \geq 2$ are pure contrarians and an informational cascade with complete anti-herding occurs.

**Proof.** In the following, I only show that the claims hold for $\theta_n$. The full claims easily follow from $\theta_n = -\theta_n$.

part (a): It needs to be shown that $\exists n_0$ s.t. $\theta_n = -b$ and $\theta_n = b \forall n > n_0$. Plugging the inequality $E^+(-\theta_n) > -\theta_n$ into equation (4) gives $\theta_{n+1} < \theta_n - \sigma$. Hence $\theta_n = -1$ after finitely many steps. Complete herding requires that $\theta_2 = \theta_3 = -1$ which holds if $\sigma \geq b - E^+(0)$.

part (b): It needs to be shown that $\theta_n, \theta_n \in (-b, b) \forall n$. Using $E^+(\theta_n) > \theta_n$ in both equations (3) and (4) gives $\theta_{n+1} < \min\{\theta_n, -\theta_n\} - \sigma$. Since $\sigma \geq -b$ it follows that $\theta_{n+1} < b$. On the other hand, equation (4) and the inequality $E^+(-\theta_n) < b$ imply $\theta_{n+1} > -b - \sigma \geq -b$ (since $\sigma \leq 0$). Hence, $|\theta_n| < b \forall n$ which implies the impossibility of an informational cascade. See Çelen and Kariv (2004) for the proof in the special case $\sigma = 0$. In case $\sigma = -E^+(0)$, it is easy to check that $\theta_n = 0$ implies $\theta_{n+1} = -E^+(0) - \sigma = 0$ (since $F(0) = 1/2$). Hence, the claim follows by induction since $\theta_1 = 0$.

part (c): It needs to be shown that $\theta_n > 0$ and $\theta_n < 0$. This follows from part (b) since $\theta_n > -b - \sigma > 0$. Moreover, it immediately follows from equation (3) that $\theta_{n+1} > b$ for all $n > 1$ provided that $|\sigma|$ is sufficiently large. \qed
4 Binary Signals

Let $N = \infty$. The space of states of the world as well as the space of agents’ private signals are binary, i.e. $\Theta \equiv \Theta_\mathcal{S} \equiv \{-1, 1\}$. The prior distribution of $\theta_0$ is symmetric and common knowledge. The precision of each agent $n$’s signal satisfies $\Pr(\theta_n = 1 | \theta_0 = 1) = \Pr(\theta_n = -1 | \theta_0 = -1) = q > 0.5$.

Assuming perfect observability, i.e. $\mathcal{I}_n = \mathcal{I}_n^{PO}$, the optimality condition (1) becomes:

$$a_n = 1 \text{ iff } \mathbb{E}(\theta_0 | \mathcal{I}_n^{PO}) \geq -\sigma \sum_{i=1}^{n-1} a_i.$$  

Let $p_n$ denote agent $n$’s (private) belief about the likelihood of the event $\theta_0 = 1$, conditional on her information $\mathcal{I}_n^{PO}$. Hence, $\mathbb{E}(\theta_0 | \mathcal{I}_n^{PO}) = 2p_n - 1$ and the optimal decision rule of agent $n$ can be rewritten in the following way:

$$a_n = 1 \text{ iff } p_n \geq \hat{p}_n,$$

where $\hat{p}_n$ is a stochastic process which satisfies $\hat{p}_1 = 0$. This process specifies cutoff values of agent’s private beliefs and is therefore referred to as the cutoff process. Furthermore, let $b_n$ denote agent $n$’s belief before she observes $\theta_n$ and let $b_{\theta_n}^n$ denote her belief after having observed $\theta_n$, i.e. $p_n = b_{\theta_n}^n$. Bayesian updating implies $b_{\theta_n}^1 = \frac{q b_n}{q b_n + (1-q)(1-b_n)}$ and $b_{\theta_n}^{-1} = \frac{(1-q)b_n}{(1-q)b_n + q(1-b_n)}$. I refer to $\{b_n\}_{n \geq 1}$ as the process of public beliefs and I denote its state space by $B = \{B_i\}_{i=-\infty}^{+\infty}$, i.e.

$$B_i = \left( 1 + \left( \frac{1 - q}{q} \right)^i \right)^{-1}.$$  

Let me first characterise the behavior of the first two agents before proceeding to a more general analysis. Agent one follows her private signal, i.e. $a_1 = \theta_1$, since $\hat{p}_1 = 0.5$. For the sake of illustration, assume that $a_1 = \theta_1 = 1$ (in the other case, the inverse reasoning holds). Agent two observes $a_1$, correctly deduces $\theta_1$ and updates her belief via Bayes’ rule to $b_2 = q$. Upon observing her private signal and applying Bayes’ rule for a second time, agent two’s private belief either becomes $p_2 = b_2^1 = q^2/(q^2 + (1-q)^2)$ (if $\theta_2 = 1$) or $p_2 = b_2^{-1} = 0.5$ (if $\theta_2 = -1$). Her optimal behavior as a function of $q$ and $\sigma$ can accordingly be derived from her cutoff-value $\hat{p}_2 = 0.5(1 - \sigma)$. In case
θ_2 = 1, one obtains a_2 = 1 iff \( \sigma \geq 1 - 2q^2/(q^2 + (1 - q)^2) \) and in case \( \theta_2 = -1 \), one obtains \( a_2 = 1 \) iff \( \sigma \geq 0 \). Figure 6 summarizes agent two’s behavior for different combinations of \( \sigma \) and \( q \). Note that if \( \theta_2 = -\theta_1 \), signals cancel each other out and the optimal choice of agent two depends on whether strategic externalities are positive or negative. Hence, if \( \sigma > 0 \), agent two always imitates the action of agent one, irrespectively of her own signal (i.e. she is a pure conformist in the spirit of section 4). If, however, \( \sigma \) is negative and agent two observes a private signal which is equal to the action of agent one, i.e. \( \theta_2 = a_1 \), her optimal choice depends on the tradeoff between the positive informational externality of two signals in favor of action one against the strategic substitutability induced by agent one’s action. Hence, this tradeoff depends on the relative magnitudes of \( q \) and \( |\sigma| \). For any \( q \), agent two chooses the action opposition to agent one’s action, provided that \( \sigma \) is sufficiently small (i.e. she is a pure contrarian in the spirit of section 4). It is easy to check that the parameter combinations under which agent two reveals her private information, i.e. under which she is purely self-reliant, is given by \( \{(\sigma, q) : 1 - 2q^2/(q^2 + (1 - q)^2) < \sigma < 0 \} \), see figure 6. In the following, I analyze the learning process more generally, by studying the behavior of agents beyond agent two.

![Figure 6: For different combinations of q and σ, agent two is either a pure conformist, purely self-reliant or a pure contrarian.](image)

First, consider the case \( \sigma > 0 \).\(^3\) Agent one chooses the action that equals her signal, i.e. \( a_1 = \theta_1 \) and agent two chooses \( a_2 = a_1 \), since strategic complementarities provide a strict incentive to imitate (see above). Hence, agent two does not reveal her private information. Consequently, agent three

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\(^3\)I only study the two cases \( \sigma > 0 \) and \( \sigma < 0 \). For the case \( \sigma = 0 \) see standard models such as Banerjee (1992) and Bikhchandani et al. (1992).
finds herself in the same position as agent two and it follows by induction that no subsequent agent ever reveals her private signal. The following proposition summarizes this result.

**Proposition 10.** If \( \sigma > 0 \), only agent one reveals her private information and all subsequent agents follow her action. In other words, there exists an informational cascade with herding.

In the case of strategic complementarities, the learning process 'gets stuck' in a herd after the first agent. If \( \sigma < 0 \), a herd can not occur since negative externalities reduce the expected payoff of every new agent who joins an ongoing run and eventually the contrary action yields a higher expected payoff. Specifically, learning happens in two distinct ways. Either by some agent who breaks away from the uniform behavior of her predecessors and thereby reveals a contrary signal, or by an agent who deliberately imitates her predecessors despite the negative payoff effects of their uniform behavior and thereby reveals a signal that favors their actions. Hence, learning is necessarily more accurate in the case of strategic substitutabilities because it does not stop after agent one. However, I will show that in this case, an informational cascade occurs after some finite (and random) point in time and behavior settles down in an anti-herd. The following lemma characterizes the corresponding limit points of learning in terms of the public belief process \( \{b_n\}_n \).

**Lemma 1.** Let \( \sigma < 0 \). There exist two absorbing states \( b_{-1}^*, b_1^* \in (0, 1) \) of \( \{b_n\}_n \) that satisfy \( b_{-1}^* + b_1^* = 1 \) and \( b_{-1}^* < b_1^* \). \( \{b_n\}_n \) does not reach any other absorbing states.

**Proof.** Define \( b_1^* = \min\{ b \in B : b^{-1} > 0.5 + j \frac{|\sigma|}{2} \text{ and } b^1 < 0.5 + (j + 1) \frac{|\sigma|}{2} \} \) and \( b_{-1}^* = 1 - b_1^* \). Note that \( b_{-1}^* \) and \( b_1^* \) clearly exist since \( |b^{-1} - b^1| \) goes to 0 as \( b \) approaches either 0 or 1 and since \( |\sigma| \) is constant. Moreover, from equation (6) it is easy to check that \( B_i - 0.5 = 0.5 - B_{-i} \) for all \( i > 0 \), which implies that \( b_{-1}^* + b_1^* = 1 \) and \( b_{-1}^* < b_1^* \). In order to show that \( b_{-1}^* \) and \( b_1^* \) are absorbing states of \( \{b_n\}_n \) let \( b_n = b_{-1}^* \) or \( b_n = b_1^* \). Since, by definition, there is no \( j \in \mathbb{Z} \) that satisfies \( 0.5 + j \frac{|\sigma|}{2} \in (b^{-1}, b^1) \), it follows that \( \tilde{p}_n \notin (b^{-1}, b^1) \) and therefore the private signals of agent \( n \) and any of her successors will not be revealed. Hence, \( \{b_n\}_n \) remains constant forever and \( \sigma < 0 \) implies that actions satisfy \( a_n = -a_{n+1} \). Finally, by definition of \( b_1^* \), none of the states \( b \in (b_{-1}^*, b_1^*) \) are absorbing. \( \square \)

The two plots in figure 7 illustrate convergence to the absorbing state \( b_1^* \). Once this state is reached, an informational cascade with anti-herding occurs
(as shown by the proof of lemma 1). The cutoff-process \( \{\hat{p}_n\}_n \) fluctuates around the interval \((b_n^{-1}, b_n^1)\) but \( b_n \) is constant since no agent reveals her private information.

Lemma 1 states that there are two possible outcome of social learning, only one of which is close to the true state of the world \( \theta_0 \). Even though \( \theta_0 \) is never learned with certainty, I show below (lemma 3) that for certain parameter configurations, the public belief will eventually be very close to either one or zero. However, this does not exclude the possibility that social learning settles in near the ‘wrong’ state.

Definition 5 defines the notion of correct learning and lemma 2 provides two sufficient conditions for the probability of correct learning to be close to one.

**Definition 5.** Learning is **correct** if \( \{b_n\} \) reaches the absorbing state \( b^*_0 \). \( P_{\sigma,q} \) denotes the ex ante probability that learning is correct.

**Lemma 2.** Let \( \sigma < 0 \) and consider any \( \varepsilon > 0 \). If \( q > 0.5 \) is fixed, \( P_{\sigma,q} > 1 - \varepsilon \) if \( |\sigma| \) is sufficiently small. The same is true if \( \sigma \) is fixed and \( q \) is sufficiently close to one.

**Proof.** To keep notation simple, I assume that \( \theta_0 = 1 \) (the proof in the other case is symmetric). Let me first prove that for all \( b_n \in (b_n^*, b_n^1) \) and \( \hat{p}_n \), there exists some finite integer \( \delta \geq 1 \) such that \( \Pr(b_{n+\delta} = b^1_n) = q \), \( \Pr(b_{n+\delta} = b_n^{-1}) = 1 - q \) and \( b_{n+i} = b_n \) for all \( i = 0, \ldots, \delta - 1 \). The claim follows immediately if \( \hat{p}_n \in [b_n^{-1}, b_n^1] \) (where \( \delta = 1 \)) because in this case agent \( n \) chooses the action that equals her signal. Furthermore, let me show that if the condition \( \hat{p}_n \in [b_n^{-1}, b_n^1] \) is not satisfied, it will nevertheless hold after a finite and deterministic number of steps. Assume that \( b_n > 0.5 \) (the proof for the case \( b_n < 0.5 \) is symmetric) and let \( \hat{p}_n \notin [b_n^{-1}, b_n^1] \). Agent \( n \) does not reveal her action and therefore \( b_{n+1} = b_n \). Moreover, \( \hat{p}_{n+1} \) approaches the interval \([b_n^{-1}, b_n^1] \) by a fixed step size of \(|\sigma|/2 \) (see equation 5). Since \( b_n \in (b_n^*, b_n^1) \) implies that there exists some \( j > 0 \) such that \( 0.5 + j\frac{|\sigma|}{2} \in [b_n^{-1}, b_n^1] \) and since the step size is constant, it follows by iteration that \( \{\hat{p}_n\} \) eventually (after \( \delta \) steps) enters the interval \([b_n^{-1}, b_n^1] \). Depending on the signal of agent \( n + \delta \), either \( b_{n+\delta} = b_n^1 \) (with probability \( q \)) or \( b_{n+\delta} = b_n^{-1} \) (with probability \( 1 - q \)).

Let \( k^* \) denote the positive integer that satisfies \( b_k^1 = B_{k^*} \). It is easy to check that \( k^* \to \infty \) if either \( q \to 1 \) (for fixed \( \sigma \)) or \( |\sigma| \to 0 \) (for fixed \( q \)), since \( B_i - B_{i-1} \) monotonically approaches zero (in \( i \)) and since \( B_i \to 1 \) as \( q \to 1 \) \( \forall i > 0 \), resp. \( B_i \to 0 \) \( \forall i < 0 \). Let the random sequence \( T = \{n_1, n_2, \ldots, n^*\} \) denote the positions at which the public belief changes (i.e. \( n \in T \implies \hat{p}_{n-1} \in (b_n^{-1}, b_n^1) \)). The first part of the proof implies that all \( n_{i+1} - n_i \) are finite. Let
Figure 7: Two simulation runs of the processes \( \{b_n\} \) (black), \( \{\hat{p}_n\} \) (dark grey), \( \{b_{n-1}\} \) (light grey) and \( \{b_1\} \) (light grey) for \( \sigma = -0.1 \) (first plot) and \( \sigma = -0.02 \) (second plot).
me define a stochastic process \( \{B_t\}_{t=1}^\infty \) by

\[
B_t = \begin{cases} 
  b_n & \text{if } t \leq n^* \\
  B_{t-1} & \text{if } t > n^*
\end{cases}
\]

\( \{B_t\} \) is a simple Markov Process with finite state space \( B \cap [b_{n^*-1}, b_{n^*}] \) and transition probabilities \( \Pr(B_t \to B_{t+1}) = 1 - \Pr(B_t \to B_{t-1}) = q \) and \( \Pr(b^n \to b^{n+1}) = \Pr(b^n_{n^*-1} \to b^n_{n^*}) = 1 \). Hence, the problem is equivalent to the ‘Gambler’s Ruin Problem’ which states that \( P_{\sigma,q} = \Pr(\exists t > 0 : B_t = b^*_1) = (1 + m k^*)^{-1} \), where \( m = (1 - q)/q \). Hence, \( P_{\sigma,q} \to 1 \) is implied by either \( q \to 1 \) or \( |\sigma| \to 0 \).

The intuition of the proof of lemma 2 is the following. The social learning process can be rewritten as a Markov process which ‘jumps’ towards the correct state with probability \( q \) and away from it with probability \( 1 - q \). If \( q \) is large, the outcome of the process is therefore more likely to be correct. On the other hand, the smaller \( |\sigma| \), the more agents find it optimal to herd on the same action. In other words, it takes more steps for the learning process to reach one of the absorbing states. This implies that learning is correct with higher probability (the solution to the problem of convergence is equivalent to the solution of a special version of the Gambler’s Ruin Problem\(^4\)).

As noted before, an informational cascade always occurs after finite time. However, the following proposition implies that the learning process may nevertheless end up very close to either zero or one. This is the case because the expected start of a cascade goes to infinity as \( |\sigma| \) tends to zero. The lemma establishes this result via a simple relation between the precision of learning and the probability of correct learning.

**Lemma 3.** \( b^*_{\sigma_0} = P_{\sigma,q} \) and \( b^-_{\sigma_0} = 1 - P_{\sigma,q} \).

**Proof.** Let \( k^* \) denote the positive integer that satisfies \( b^*_1 = B_k^* \) and let \( m = (1 - q)/q \). It is a well known fact that \( P_{\sigma,q} = (1 - m k^*)/(1 - m 2 k^*) \), provided that \( q \neq 0.5 \) (this is the solution to a version of the Gambler’s Ruin Problem in which the gambler seeks to double her fortune, cf. footnote 4). From the fact that the denominator can be rewritten as \( (1 - m k^*) \cdot (1 + m k^*) \), it easily follows that \( P_{\sigma,q} = B_{k^*} = 1/(1 + m k^*) \).

The following proposition summarizes the results of this section.

---

\(^4\)A special version of the Gambler’s Ruin Problem can be described as follows. A gambler is offered an (infinite) sequence of bets. In each bet she gains one unit of money with probability \( q \) and loses one unit with probability \( 1 - q \). Starting out with a fortune of size \( M \), one can determine the probability that her fortune reaches \( 2M \) before she goes broke.
Proposition 11. If $\sigma < 0$, the outcome of social learning is arbitrarily close to the true state of the world with arbitrarily high probability, provided that either $|\sigma|$ is sufficiently small (for given $q$) or $q$ is sufficiently close to one (for given $\sigma$).

Proof. The claim is implied by lemma 2 and lemma 3.

Proposition 11 states that in the limit as $|\sigma|$ tends to 0, the true state of the world is learned with probability one. This contrasts with the standard herding models without strategic externalities such as the ones in Banerjee (1992) and Bikhchandani et al. (1992). Table 8 shows explicit values of $b_1^* = P_{\sigma,q}$ for different parameter combinations, provided that the true state of the world satisfies $\theta_0 = 1$.

Note, however, that if $|\sigma|$ and $q$ are both small, learning may take a long time to reach one of the absorbing states. For example, in case $q = 0.51$ and $\sigma = -0.01$, one obtains an average time of convergence of around 3555 periods. Other parameter combinations produce considerably faster convergence (e.g. around 10.5 periods for $q = 0.7$ and $\sigma = -0.2$, despite a probability of convergence as high as 96.7%), see figure 9.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\sigma$</th>
<th>-0.2</th>
<th>-0.1</th>
<th>-0.05</th>
<th>-0.02</th>
<th>-0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.51</td>
<td>0.520</td>
<td>0.520</td>
<td>0.540</td>
<td>0.885</td>
<td>0.943</td>
<td></td>
</tr>
<tr>
<td>0.52</td>
<td>0.540</td>
<td>0.579</td>
<td>0.863</td>
<td>0.954</td>
<td>0.977</td>
<td></td>
</tr>
<tr>
<td>0.55</td>
<td>0.833</td>
<td>0.917</td>
<td>0.961</td>
<td>0.985</td>
<td>0.992</td>
<td></td>
</tr>
<tr>
<td>0.60</td>
<td>0.945</td>
<td>0.975</td>
<td>0.989</td>
<td>0.995</td>
<td>0.998</td>
<td></td>
</tr>
<tr>
<td>0.70</td>
<td>0.967</td>
<td>0.986</td>
<td>0.994</td>
<td>0.997</td>
<td>0.999</td>
<td></td>
</tr>
</tbody>
</table>

Figure 8: Probabilities $P_{\sigma,q}$ of convergence to the correct belief $b_1^*$, where $P_{\sigma,q} = b_1^*$.

\footnote{These values have been determined analytically according to the procedures used in the proofs of lemma 1 and lemma 2.}

\footnote{The average has been calculated over one million simulation runs.}

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<table>
<thead>
<tr>
<th>$q$</th>
<th>$\sigma$</th>
<th>-0.2</th>
<th>-0.05</th>
<th>-0.01</th>
</tr>
</thead>
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<td>7.0</td>
<td>26.0</td>
<td>3554.5</td>
<td></td>
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<tr>
<td>0.55</td>
<td>39.1</td>
<td>173.2</td>
<td>577.4</td>
<td></td>
</tr>
<tr>
<td>0.70</td>
<td>10.5</td>
<td>33.4</td>
<td>147.1</td>
<td></td>
</tr>
</tbody>
</table>

Figure 9: Average time of reaching the correct absorbing state $b_1^*$, conditional on reaching it (1 million simulation runs each).
Appendix A: Continuous State Space, Binary Signals, Complete Learning and the Rule of Succession

In this appendix, I study a simple model which is different from the basic model of section 2. It also incorporates the idea of learning in the presence of strategic externalities (negative externalities in this case) but unlike the previous model, agent’s payoffs do not only depend on predecessors’ actions, but on the actions of all agents.

Let there be \( N \) firms, indexed by \( n = 1, 2, ..., N \). Each firm has to make a once-in-a-lifetime entry decision \( a_n \) between two markets \( A \) and \( B \). Total profits in both markets are known to be of size one, but the exact distribution is unknown. Let \( \theta_0 \) denote the fraction of total profits that are earned in market \( A \). It is common knowledge that \( \theta_0 \) is drawn from a uniform distribution on \([0, 1]\). Each firm \( n \) employs an expert who makes a recommendation \( \theta_n \in \{A, B\} \), and each firm only observes their own expert’s recommendation. Moreover, recommendations are independent of each other and accurate in the sense that \( \Pr(\theta_n = A) = 1 - \Pr(\theta_n = B) = \theta_0 \). Profits in each market are divided equally among all companies who operate within this market. Let \( N_A \) denote the total number of firms who choose market \( A \). Firms are risk-neutral with utility function

\[
    u(a) = \begin{cases} 
      \frac{\theta_0 N_A}{N - N_A} & \text{if } a = A \\
      \frac{1 - \theta_0}{N - N_A} & \text{if } a = B 
    \end{cases}
\]

Decisions are taken sequentially, where each firm \( n \) observes the actions of a subset of size \( \lambda_n \in \{0, 1, ..., n - 1\} \) of its predecessors.\(^7\) Moreover, firms update their beliefs about \( \theta_0 \) via Bayes’ rule. The following proposition shows that there always exists an efficient equilibrium in which all firms are purely self-reliant (i.e. they entirely rely on their own information and neglect the actions of others). This result is independent of the information structure. The reason why the information about the behavior of others is worthless is that for each agent’s action choice, the strategic substitutability induced by the action exactly outweighs the corresponding positive informational externality.

**Proposition 12.** There exists an equilibrium in which all firms reveal their private information, i.e. \( a_n = \theta_n \) for all \( n \), regardless of their sample size \( \lambda_n \).

\(^7\)It is irrelevant to the solution whether the information structure of the firms are known by the other firms.
Proof. Consider any firm $n$ and assume that all other firms $n' \neq n$ play according to the strategy $a_{n'} = \theta_{n'}$. I show that agent $n$’s best response to this strategy profile is $a_n = \theta_n$.

Let $k \leq \lambda_n$ denote the number of predecessors in firm $n$’s sample who chose actions $A$ and let $\tilde{N}_A$ denote the total number of agents other than agent $n$ who choose action $A$. Hence,

$$E[u(A)|\theta_n] = \frac{N-1-\lambda_n}{N+2} \sum_{i=0}^{N-1-\lambda_n} \Pr[\tilde{N}_A = k+i|\theta_n] \frac{1+k+i+1_{\{\theta_n=A\}}}{1+k+i}$$

(7)

It follows from the uniform prior and Bayesian updating that, conditional on $\tilde{N}_A$ and $\theta_n$, $\theta_0$ follows a Beta distribution with parameters $\alpha = 1 + \tilde{N}_A + 1_{\{\theta_n=A\}}$ and $\beta = 1 + N - \tilde{N}_A - 1_{\{\theta_n=A\}}$ (cf. the rule of succession). Since the expected value of a Beta distributed random variable with parameters $\alpha$ and $\beta$ is $\alpha/ (\alpha + \beta)$, it follows that:

$$E[u(A)|\theta_n] = \frac{1}{N+2} \sum_{i=0}^{N-1-\lambda_n} \Pr[\tilde{N}_A = k+i|\theta_n] \frac{1+k+i+1_{\{\theta_n=A\}}}{1+k+i}.$$ 

Similarly,

$$E[u(B)|\theta_n] = \frac{1}{N+2} \sum_{i=0}^{N-1-\lambda_n} \Pr[\tilde{N}_A = k+i|\theta_n] \frac{N-k-i-1_{\{\theta_n=A\}}}{N-k-i}.$$ 

The claim follows from pairwise evaluation of the summands of these two expressions. If $\theta_n = A$, one gets

$$\frac{2+k+i}{1+k+i} > \frac{N-k-i}{N-k-i} = 1,$$

and if $\theta_n = B$, one gets

$$\frac{1+k+i}{1+k+i} = 1 < \frac{N-k-i+1}{N-k-i}.$$

Hence, $E[u(A)|A] > E[u(B)|A]$ and $E[u(A)|B] < E[u(B)|B]$ are satisfied regardless of the actions in agent $n$’s sample. This implies that firm $n$’s best-response to the strategy profile is given by $a_n = \theta_n$. 

\[ 29 \]
Appendix B

Proof of proposition 3

Assume that $\theta_n'$ is always within the interval $[-1, 1]$, i.e. $\theta_n = \theta_n'$. The case in which $\theta_n'$ 'jumps' out of the interval is mathematically straightforward and challenging only in terms of the notation.

First, let me show by induction that $\Pr(a_n = 1) = \frac{1}{2}$ and $\theta_n + \theta_n = 0$ for all $n$. Assume that $\Pr(a_m = 1) = \Pr(a_m = -1) = \frac{1}{2}$ and $\theta_{m+1} + \theta_{n+1} = 0$ is satisfied for some $m$ and let me show that this implies that it must be satisfied for $m+1$ as well. Symmetry implies directly that $\Pr(a_{m+1} = 1) = \Pr(a_{m+1} = -1) = \frac{1}{2}$. Moreover,

\[
\theta_{m+2} + \theta_{m+2} = -E(\theta_{m+1} | a_{m+1} = 1) - E(\theta_{m+1} | a_{m+1} = -1) - \sum_{j \in \{-1,1\}} \sum_{k \in \{-1,1\}} \Pr(a_m = j | a_{m+1} = k) E \left[ \sum_{i=1}^{m-2} (\theta_i + \sigma a_i) | a_m = j, a_{m+1} = k \right] = \theta_{m-1} (\Pr(a_m = 1 | a_{m+1} = 1) - \Pr(a_m = -1 | a_{m+1} = 1) + \Pr(a_m = 1 | a_{m+1} = -1) - \Pr(a_m = -1 | a_{m+1} = -1)) = 0.
\]

The last equality is implied by the fact that the second and the third (as well as the first and the fourth) summand within the brackets add up to zero (this is easy to proof). It remains to show that the statement holds for $m = 1$. Clearly, $\Pr(a_1 = 1) = \Pr(\theta_1 \geq 0) = 0.5$ and straightforward evaluation of equation (2) in section 3 yields $\theta_2 = - (\sigma + 1/2)$ and $\theta_2 = \sigma + 1/2$.

Moreover, equation (2) in section 3 implies that

\[
\theta_n = -E \left[ \sum_{i=1}^{n-2} (\theta_i + \sigma a_i) | a_{n-1} = 1 \right] - E \left[ \theta_{n-1} + \sigma a_{n-1} | a_{n-1} = 1 \right]
\]
Let me calculate the two parts of the right-hand expression separately.

\[
E \left[ \sum_{i=1}^{n-2} (\theta_i + \sigma a_i) \mid a_{n-1} = 1 \right]
\]

\[
= \sum_{k \in \{-1, 1\}} \Pr(a_{n-2} = k \mid a_{n-1} = 1) E \left[ \sum_{i=1}^{n-2} (\theta_i + \sigma a_i) \mid a_{n-1} = 1, a_{n-2} = k \right]
\]

\[
= \frac{\Pr(a_{n-2} = 1) \Pr(a_{n-1} = 1 \mid a_{n-2} = 1)}{\Pr(a_{n-1} = 1)} (-\overline{\theta}_{n-1})
\]

\[
+ \frac{\Pr(a_{n-2} = -1) \Pr(a_{n-1} = 1 \mid a_{n-2} = -1)}{\Pr(a_{n-1} = 1)} (-\overline{\theta}_{n-1})
\]

\[
= \frac{\frac{1}{4}(1 - \overline{\theta}_{n-1})}{\frac{1}{4}(1 - \overline{\theta}_{n-1}) + \frac{1}{4}(1 + \overline{\theta}_{n-1})} (-\overline{\theta}_{n-1}) + \frac{\frac{1}{4}(1 + \overline{\theta}_{n-1})}{\frac{1}{4}(1 - \overline{\theta}_{n-1}) + \frac{1}{4}(1 + \overline{\theta}_{n-1})} (+\overline{\theta}_{n-1})
\]

\[
= \frac{1 - \overline{\theta}_{n-1}}{2} (-\overline{\theta}_{n-1}) + \frac{1 + \overline{\theta}_{n-1}}{2} (+\overline{\theta}_{n-1}) = \overline{\theta}_{n-1}^2
\]

Furthermore,

\[
E [\theta_{n-1} \mid a_{n-1} = 1] = \sum_{k \in \{-1, 1\}} \Pr(a_{n-2} = k \mid a_{n-1} = 1) E [\theta_{n-1} \mid a_{n-2} = k]
\]

\[
= (1 - \overline{\theta}_{n-1}) \frac{1 + \overline{\theta}_{n-1}}{2} + (1 + \overline{\theta}_{n-1}) \frac{1 - \overline{\theta}_{n-1}}{2} = \frac{1}{2} \left( 1 - \overline{\theta}_{n-1}^2 \right)
\]

Hence,

\[
\overline{\theta}_n = -\overline{\theta}_{n-1}^2 - \frac{1}{2} \left( 1 - \overline{\theta}_{n-1}^2 \right) - \sigma = -\sigma - \frac{1}{2} \left( 1 + \overline{\theta}_{n-1}^2 \right)
\]

and \( \overline{\sigma}_n = -\overline{\theta}_n = \sigma + \frac{1}{2} \left( 1 + \overline{\theta}_{n-1}^2 \right) \).

Q.E.D.
References


